Bounding the Distance of a Controllable System

to an Uncontrollable one

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ABSTRACT.- Let (A,B) be a pair of matrices representing a time-invariant linear system $\dot{x}(t)=Ax(t)+Bu(t)$ under block-similarity equivalence.

In this paper we measure the distance between a controllable pair of matrices (A,B) and the nearest uncontrollable one.

A bound is obtained in terms of singular values of the controllability matrix $C(A,B)$ associated to the pair. This bound is not simply based on the smallest singular value of $C(A,B)$ contrary to what one may expect.

Also a lower bound is obtained using geometrical techniques expressed in terms of the singular values of a matrix representing the tangent space of the orbit of the pair (A,B).

INTRODUCTION

We consider pairs of matrices $(A,B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ corresponding to a time-invariant linear systems $\dot{x}(t) = Ax(t) + Bu(t)$. For convenience, we identify the pair (A,B) with the rectangular block-matrix $(A \quad B)$. We consider the following action of the state feedback group $\mathcal{G} \subset Gl(n+m;\mathbf{C})$, according to the formula

$$\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \cdot (A,B) = P^{-1}(A,B) \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

The controllability matrix is defined as

$$C(A,B) = (\, B \quad AB \quad \dots \quad A^{n-1}B \,)$$

and it is well known that the pair of matrices (A,B) is controllable if and only if $\operatorname{rank} C(A,B) = n$.

We consider the set $\mathcal{C} = \{(A,B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C}); (A,B)\,\text{controllable}\}$. This is an open dense set in the space of all pairs of matrices $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ and it is invariant with respect to the $\mathcal{G}$-action.

For each $(A,B) \in \mathcal{C}$ there exists an open neigbourhood of (A,B) relatively small, such that all pairs of matrices in it are controllable. Then it makes sense to consider the distance to the nearest uncontrollable one, and to deduce safety neighbourhoods for controllable pairs of matrices.

One of the main goals of this paper is to show that a bound of this distance (in both real and complex cases) can be obtained. The method used for that, as in [**1**], is to explore the singular values of the controllability matrix of the pair (A,B). In [**1**] the distance of a controllable pair to the nearest uncontrollable one is measured considering the action of the general linear group via change of basis in the state space. In our case we consider feedback the action and we need firstly, to ensure that the controllability matrix is invariant under feedback equivalence. The norm considered in this case is the 2-norm

The sets of equivalent pairs under state feedback relation are differentiable manifolds called orbits. Given a controllable pair of matrices the nearest uncontrollable one remains obviously, in another orbit. Then the problem can be reduced to compute the distance from (A,B) to the orbits of uncontrollable pairs. For that we explore the singular values of a matrix representing the tangent space to the orbit of the pair (A,B). In [**4**] a lower bound given safety neigbourhoods is obtained considering matrix pencils $(A,B) + \lambda(I,0)$ under strictely equivalence. In spite of if two pairs of matrices are feedback equivalent their associate pencils are strictely equivalent, we remark that we can perturb a pair of matrices (A,B) considering it as a pencil, in such a way the perturbed pencil does not represent a pair of matrices in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$. As a consequence we improve a little the size of safety neigbourhoods.

In this case the norm considered is the Frobenius norm because it is easier to obtain the norm of the matrix representing the tangent space of a orbit.

I. PRELIMINARIES

(I.1) Equivalence relation

(I.1.1) We recall that the *state feedback group* is the subgroup of the linear group $Gl(n+m;\mathbf{C})$ consisting of the matrices of the form $\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$ where $P \in Gl(n,\mathbf{C})$, $Q \in Gl(m,\mathbf{C})$ and $R \in M_{m\times n}(\mathbf{C})$. We will denote this group by $\mathcal{G}$ and its unit element by I. We can identify $\mathcal{G}$ with the open subset $\{(P,Q,R); \det P \neq 0, \det Q \neq 0\}$ of the space of triples $M_n(\mathbf{C}) \times M_m(\mathbf{C}) \times M_{m\times n}(\mathbf{C})$, so that $\mathcal{G}$ is a complex manifold and its tangent space at the identity is $T_I\mathcal{G} = M_n(\mathbf{C}) \times M_m(\mathbf{C}) \times M_{m\times n}(\mathbf{C})$.

(I.1.2) We consider the following action of $\mathcal{G}$ on $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$,

$$\alpha : \mathcal{G} \times (M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})) \longrightarrow M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

defined by

$$\alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A,B)\right) = P^{-1}(A,B)\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

(I.1.3) The action defined by α induces the following equivalence relation between pairs of matrices: (A,B) and (C,D) are called *block-similar* if and only if there exists $\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \in \mathcal{G}$ such that $\alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A,B)\right) = (C,D)$.

Then, the manifold of pairs of matrices in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ which are equivalent to (A,B) is its orbit under the action of α and we will denote it by $\mathcal{O}(A,B)$.

We recall that a pair (A,B) is structurally stable if and only if there exists a neigbourhood of this pair in the space of pairs of matrices formed by pairs equivalent to it. Then (A,B) is structurally stable if and only if $\dim \mathcal{O}(A,B) = n^2 + nm$. Willems in [**10**] gives a characterization of structurally pairs in terms of its structural invariants.

(I.1.4) There exists a canonical reduced form, called Kronecker canonical form, representing each equivalence class under the equivalence considered in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$. If (A,B) is in the Kronecker canonical form, it has the following form (see [**7**], Theorem (6.2.5) for more details)

$$A = \begin{pmatrix} N & 0 \\ 0 & J \end{pmatrix} \qquad B = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

where

a) $N = \mathrm{diag}(N_1, \ldots, N_{r_0})$ with $N_i = \begin{pmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{pmatrix} \in M_{k_i}(\mathbf{C})$, $1 \leq i \leq r_0$.

We assume that $k_1 \geq \ldots \geq k_{r_0}$, and we write $p = k_1 + \cdots + k_{r_0}$.

b) Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues.

$J = \mathrm{diag}(J_1, \ldots, J_s)$, $J_i = \mathrm{diag}(J_{\delta_1(i)}, J_{\delta_2(i)}, \ldots)$ with

$$J_{\delta_j(i)} = \lambda_i I_{\delta_j(i)} + \begin{pmatrix} 0 & I_{\delta_j(i)-1} \\ 0 & 0 \end{pmatrix} \in M_{\delta_j(i)}(\mathbf{C}),\ 1 \le i \le s,\ j = 1, 2, \dots .$$

We assume that for each i, $\delta_1(i) \ge \delta_2(i) \ge \dots$ and we have $\sum_{i,j} \delta_j(i) = n - p$.

c) $E = \operatorname{diag}(E_1, \dots, E_{r_0})$ with $E_i = (0, \dots, 0, 1)^t \in M_{k_i \times 1}(\mathbf{C})$, $1 \le i \le r_0$.

Notice that $(N, (E\ 0))$ is a controllable pair of matrices in $M_p(\mathbf{C}) \times M_{p\times m}(\mathbf{C})$ and it is called the controllable pair of (A, B).

(I.2) Controllability matrix

(I.2.1) The controllability matrix of a pair $(A, B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ is defined as

$$C(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}.$$

PROPOSITION. *The rank of the controllability matrix is invariant under the equivalence relation considered.*

In fact this result is more general:

PROPOSITION. *Given a pair $(A, B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$, the scalars*

$$\operatorname{rank} \begin{pmatrix} B & AB & \dots & A^i B \end{pmatrix}, \quad \forall i \ge 0$$

are invariant under feedback equivalence.

PROOF: Let (A, B) and (A_1, B_1) equivalent pairs. Then there exist invertible matrices P and Q and a rectangular matrix R such that $(A_1, B_1) = (P^{-1}AP + P^{-1}BR, P^{-1}BQ)$. So

$$\begin{pmatrix} P^{-1} & & \\ & \ddots & \\ & & P^{-1} \end{pmatrix} \begin{pmatrix} A & B & & & & \\ I & 0 & A & B & & \\ & & I & 0 & & \\ & & & & \ddots & \\ & & & & & A\ \ B \\ & & & & & I\ \ 0 \end{pmatrix} \begin{pmatrix} P & 0 & & & & \\ R & Q & & & & \\ & & P & 0 & & \\ & & R & Q & & \\ & & & & \ddots & \\ & & & & & P\ \ 0 \\ & & & & & R\ \ Q \end{pmatrix} =$$

$$= \begin{pmatrix} A_1 & B_1 & & & & \\ I & 0 & A_1 & B_1 & & \\ & & I & 0 & & \\ & & & & \ddots & \\ & & & & & A_1\ \ B_1 \\ & & & & & I\ \ \ 0 \end{pmatrix}$$

And for all (A, B) we have

$$\operatorname{rank} \begin{pmatrix} A & B & & & & \\ I & 0 & A & B & & \\ & & I & 0 & & \\ & & & & \ddots & \\ & & & & & A\ \ B \\ & & & & & I\ \ 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} B\ \ AB\ \dots\ A^iB & & & & \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{pmatrix}$$

Another proof of this proposition can be deduced as a corollary of the result given in ([**7**], Lemma 6.3.1) where compute the dimensions of $\sum_{i=0}^{s} \operatorname{Im} A^iB$ and $\sum_{i=0}^{s} \operatorname{Im}(A+BK)^iB$ for $s = 0, \dots, n$ and any $m \times n$-matrix K.

(I.2.2) The scalars

$$\begin{aligned}
\rho_{-1} &= 0 \\
\rho_0 &= \operatorname{rank}(B) \\
\rho_1 &= \operatorname{rank}\begin{pmatrix} B & AB \end{pmatrix} \\
&\vdots \\
\rho_i &= \operatorname{rank}\begin{pmatrix} B & AB & \dots & A^iB \end{pmatrix}
\end{aligned}$$

characterize the controllable part of each pair of matrices in the following manner. We consider the conjugate partition $[k_1, \dots, k_{r_0}]$ of

$$\{r_k = \rho_k - \rho_{k-1}\}_{0 \le k \le n}.$$

The scalars k_i are called controllability indices.

It is well know the following characterization of controllable pairs of matrices.

PROPOSITION. *A pair of matrices $(A, B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ is controllable if and only if the controllability matrix has full rank, i.e.*

$$\operatorname{rank} C(A, B) = n.$$

Notice that, if (A, B) is a controllable pair, $k_1 + \ldots + k_{r_0} = n$.

It is interesting to remark that if (A, B) is structurally stable then it is controllable. The converse it is only true in the case where $m = 1$.

We consider the set $\mathcal{C} = \{(A, B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C});\ (A, B) \text{ controllable}\}$. Taking into account the upper semicontinuity of rank, any small perturbation of a matrix increase the rank. Then we have that $\mathcal{C}$ is a dense set in the space of all pairs of matrices $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$. In the other hand if the matrix $C(A, B)$ has full rank it is in a neigbourhood then the set is also an open set.

(I.4) Strata

The understanding of which orbits that are close to an orbit is revealed by the stratification of space of pairs of matrices. Then we define the strata.

(I.4.1) A *stratum*, $E(\sigma)$, in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ consists of all pairs of matrices having the same collection of discrete invariants than (A, B), $\sigma = \{(k_1, \dots, k_{r_0}); (\delta_1(1), \dots, \delta_{l_1}(1)), \dots, (\delta_1(s), \dots, \delta_{l_s}(s))\}$. We denote by $E(A, B)$ the stratum of the pair (A, B).

Then, there are only finitely many strata, partitioning $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$. Each one is an orbit or an uncountable union of block-similarity equivalence classes, differing only in the values of the eigenvalues $\lambda_1, \dots, \lambda_s$.

García-Planas in [**5**], proved that any strata is a differentiable manifold which dimension is given by $\dim E(A, B) = s + \dim \mathcal{O}(A, B)$ where s is the number of distinct eigenvalues of (A,B). Also is proved that they verify the frontier condition, that is to say, the boundary of any stratum is formed by strata of strictely lower dimension. In [**8**] a characterization of $\overline{\mathcal{O}(A,B)} \supset \overline{\mathcal{O}(C,D)}$ in terms of the structural invariants is given.

We present here as an example, the hierarchic closure of the set of 3×2 pairs of matrices.

Stratification of 3×2 *pairs of matrices*

$$
\begin{array}{ccc}
 & \left(\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}\right) & \text{codim } 0\\
 \swarrow & \downarrow & \\
\left(\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix}\right); & \left(\begin{pmatrix}0&0&0\\0&0&0\\0&0&a_1\end{pmatrix},\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\right) & \text{codim } 2\\
 \searrow & \downarrow & \\
 & \left(\begin{pmatrix}0&1&0\\0&0&0\\0&0&a_1\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix}\right) & \text{codim } 3\\
 & \downarrow & \\
 & \left(\begin{pmatrix}0&0&0\\0&a_1&0\\0&0&a_2\end{pmatrix},\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 4\\
 & \downarrow & \\
 & \left(\begin{pmatrix}0&0&0\\0&a_1&1\\0&0&a_1\end{pmatrix},\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 5\\
 \swarrow & \downarrow & \\
\left(\begin{pmatrix}0&0&0\\0&a_1&0\\0&0&a_1\end{pmatrix},\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\right); & \left(\begin{pmatrix}a_1&0&0\\0&a_2&0\\0&0&a_3\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 6\\
 \searrow & \downarrow & \\
 & \left(\begin{pmatrix}a_1&1&0\\0&a_1&0\\0&0&a_2\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 7\\
 & \downarrow & \\
 & \left(\begin{pmatrix}a_1&1&0\\0&a_1&1\\0&0&a_1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 8\\
 \swarrow & \downarrow & \\
\left(\begin{pmatrix}a_1&0&0\\0&a_1&0\\0&0&a_2\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right); & \left(\begin{pmatrix}a_1&1&0\\0&a_1&0\\0&0&a_1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 9\\
 \searrow & \downarrow & \\
 & \left(\begin{pmatrix}a_1&0&0\\0&a_1&0\\0&0&a_1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right) & \text{codim } 12
\end{array}
$$

(I.5) The tangent and normal spaces to the orbit

(I.5.1) Let (A,B) be a pair of matrices in $M_n(\mathbf{C})\times M_{n\times m}(\mathbf{C})$. It is not dificult to check that the tangent space of its orbit $T_{(A,B)}\mathcal{O}(A,B)$ is given in the following manner

$$T_{(A,B)}\mathcal{O}(A,B) = \{(X,Y) = ([A,P] + BR, BQ - PB);\ \left(\begin{smallmatrix} P & 0 \\ R & Q \end{smallmatrix}\right) \in T_I\mathcal{G}\}.$$

Using the Kronecker products and vec-operator (see [**9**] for their definition and properties), we can represent the $n^2 + nm$ vectors $(X,Y) \in T_{(A,B)}\mathcal{O}(A,B)$ in the form

$$\begin{pmatrix} \mathrm{vec}\,(X) \\ \mathrm{vec}\,(Y) \end{pmatrix} = \begin{pmatrix} A \otimes I_n - I_n \otimes A^t & B \otimes I_n & 0 \\ -I_n \otimes B^t & 0 & B \otimes I_m \end{pmatrix} \begin{pmatrix} \mathrm{vec}\,(P) \\ \mathrm{vec}\,(R) \\ \mathrm{vec}\,(Q) \end{pmatrix}.$$

In this notation, we may say that the tangent space is the range of the $(n^2+nm) \times (n^2+nm+m^2)$-matrix

$$\mathbf{T} = \begin{pmatrix} A \otimes I_n - I_n \otimes A^t & B \otimes I_n & 0 \\ -I_n \otimes B^t & 0 & B \otimes I_m \end{pmatrix}.$$

Then we have the following result.

THEOREM.

$$\dim T_{(A,B)}\mathcal{O}(A,B) = \mathrm{rank} \begin{pmatrix} A \otimes I_n - I_n \otimes A^t & B \otimes I_n & 0 \\ -I_n \otimes B^t & 0 & B \otimes I_m \end{pmatrix} = \mathrm{rank}\,\mathbf{T}.$$

We may define the normal space $T_{(A,B)}\mathcal{O}(A,B)^\perp$ as the orthogonal to the tangent space $T_{(A,B)}\mathcal{O}(A,B)$. The orthogonality is defined with respect to the following usual inner product.

DEFINITION:

$$< (A_1, B_1), (A_2, B_2) >= \mathrm{trace}\,(A_1 A_2^* + B_1 B_2^*).$$

Obviously, we have the following.

COROLLARY.

$$\dim T_{(A,B)}\mathcal{O}(A,B)^\perp = n^2 + nm - \mathrm{rank}\,\mathbf{T} = \dim \mathrm{Ker}\,\mathbf{T} - m^2.$$

(I.5.2) After this, we observe that we can obtain the dimension of $T_{(A,B)}\mathcal{O}(A,B)$ from the singular value decomposition (s.v.d.) of the matrix $\mathbf{T}$.

COROLLARY. *In this situation*

$$\dim T_{(A,B)}\mathcal{O}(A,B)^\perp = \textit{number of zero singular values of } \mathbf{T}.$$

(I.5.3) Knowing the Kronecker structure of (A,B), it is possible to give the dimension of the orbit in terms of the collection of invariants of the pair.

PROPOSITION ([**5**]). *The dimension of each orbit is given by*

$$\dim T_{(A,B)}\mathcal{O}(A,B) = (n^2+nm) - \left(\sum_{1\le i,j\le r_0} \max\{0, k_j - k_i - 1\} + r_0(n-p) + \right.$$
$$+ (m-r_0)(p-r_0) + \sum_{1\le i\le s} (\delta_1(\lambda_i) + 3\delta_2(\lambda_i) + 5\delta_3(\lambda_i) + \ldots) +$$
$$\left. + (m-r_0)(n-p)\right) .$$

J. Demmel and A. Edelman in [**2**] give the codimension of the orbit of a pencil $\mathrm{A}+\lambda\mathrm{B}$ in terms of its discrete invariants. Notice that if we consider $\mathrm{A} = \begin{pmatrix} A & B \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} I_n & 0 \end{pmatrix}$ and counting the codimension of its orbit referred to the variety of pencils $\mathrm{A}+\lambda\mathrm{B}$ with $B = \begin{pmatrix} I_n & 0 \end{pmatrix}$ the formula presented in [**2**] coincides with this one.

(I.5.4) As a consequence, we can analyze the controllability and stability of a pair of matrices obtaining the following

PROPOSITION. *Let (A,B) be a controllable pair of matrices. Then*

$$n^2 + n + m - 1 \le \dim T_{(A,B)}\mathcal{O}(A,B) \le n^2 + nm.$$

And

$$\dim T_{(A,B)}\mathcal{O}(A,B) = n^2 + nm$$

if and only if the pair is structurally stable.

The lower bound is achieved when $r_0 = 1$.

REMARK 1: If $m = 1$, only structurally stable pairs of matrices are controllable.

REMARK 2: It is possible to find no controllable pairs of matrices (C,D) with $n^2 + nm > \dim T\mathcal{O}(A,B) = \dim T\mathcal{O}(C,D) \ge n^2 + n + m - 1$. But taking into account that the partition into strata verifies the frontier condition the pair (C,D) is not in the closure of the stratum (orbit) of the controllable pair (A,B).

We note $\dim T_{(A,B)}\mathcal{O}(A,B) = a$, with $a \ge n^2 + n + m - 1$.

II. The μ-Distance.

The generic character of $\mathcal{C}$, allows us to ensure that if $(A,B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ is a controllable pair of matrices there exists a neigborhood $\mathcal{U}$ in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ such that for all $(A_1,B_1) \in \mathcal{U}$ then (A_1,B_1) is also a controllable pair. Therefore it makes sense to consider the distance to the nearest uncontrollable pair.

DEFINITION: For a given controllable pair of matrices $(A,B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ we define the distance between (A,B) and a nearest uncontrollable pair by

$$\mu_{\mathbf{C}}(A,B) = \min_{\delta A, \delta B} \|(\delta A, \delta B)\|,$$

where $\delta A \in M_n(\mathbf{C})$ and $\delta B \in M_{n\times m}(\mathbf{C})$ such that $(A+\delta A, B+\delta B)$ is uncontrollable.

If we restrict ourselves to real perturbations we use the following

DEFINITION: For a given controllable pair of matrices $(A, B) \in M_n(\mathbf{R}) \times M_{n\times m}(\mathbf{R})$ we define the distance between (A, B) and a nearest uncontrollable pair by

$$\mu_{\mathbf{R}}(A, B) = \min_{\delta A, \delta B \text{ real}} \|(\delta A, \delta B)\|$$

where $\delta A \in M_n(\mathbf{R})$ and $\delta B \in M_{n\times m}(\mathbf{R})$ such that $(A+\delta A, B+\delta B)$ is uncontrollable.

The matrix norms considered in the follows are the 2-norm: $\|A\|_2 = \sigma_1$ where σ_1 is the largest singular value of A or the Frobenius norm: $\|A\|_F = \sqrt{a_{ij}^2}$.

R. Eising in [**3**], measured $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{R}}$ as

$$\mu_{\mathbf{C}}(A, B) = \min_{\lambda \in \mathbf{C}} \sigma_n(\lambda I - A, B)$$
$$\mu_{\mathbf{R}}(A, B) = \min_{\lambda \in \mathbf{R}} \sigma_n(\lambda I - A, B)$$

where $\sigma_n(\lambda I - A, B)$ is the smallest singular value of $(\lambda I - A, B)$. But finding the values of $\mu_{\mathbf{R}}$ can be a very involved process. However, we can give a bound of $\mu_{\mathbf{R}}$ in terms of the singular values of the controllability matrix of (A, B).

(II.1) μ-distance and controllability matrix

Now we analyze if a bound of $\|(\delta A, \delta B)\|_2$ can be deduced from the controllability matrix of a given pair of matrices (A, B).

Taking into account that the controllability is mesured by the rank controllablity matrix we need to compute the singular value decomposition (s.v.d.) of the controllability matrix $C(A, B)$ associated with a given controllable pair $(A, B) \in \mathcal{C}$.

$$C(A, B) = (\, B \quad AB \quad \dots \quad A^{n-1}B\,), \qquad A \in M_n(\mathbf{R}), B \in M_{n\times m}(\mathbf{R})$$

Calling $[\,\Sigma\,|\,0\,]$ the s.v.d. of $C(A, B)$ we have

$$C(A, B) = X^t\,[\,\Sigma\,|\,0\,]\,Y, \quad \text{where } X, Y \text{are orthogonal matrices.}$$

Let (A_1, B_1) be the pair $\alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A, B)\right)$ of matrices considering $P = X$ and $\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$ being orthogonal. Then $R = 0$ and Q is orthogonal so

$$(A_1, B_1) = (P^{-1}AP, P^{-1}BQ) = (P^tAP, P^tBQ).$$

LEMMA 1. *In the above conditions and considering the 2-norm,*

$$\mu_{\mathbf{R}}(A_1, B_1) = \mu_{\mathbf{R}}(A, B).$$

PROOF: For each $(\delta A, \delta B)$ such that $(A+\delta A, B+\delta B)$ is uncontrollable there exist $(\delta A_1, \delta B_1)$ such that $(A_1+\delta A_1, B_1+\delta B_1)$ is uncontrollable and conversely. To prove that it suffices to consider $\delta A_1 = P^{-1}\delta A P$ and $\delta B_1 = P^{-1}\delta B Q$ and make use of the invariance of the rank of the matrix $C(A,B)$ under the equivalence considered.

Then

$$\|(\delta A_1, \delta B_1)\|_2 = \|(P^{-1}(\delta A)P, P^{-1}(\delta B)Q)\|_2 = \\ = \|P^{-1}\|_2 \cdot \|(\delta A, \delta B)\|_2 \cdot \left\|\begin{pmatrix} P & \\ & Q \end{pmatrix}\right\|_2 = \|(\delta A, \delta B)\|_2 .$$

LEMMA 2. *The s.v.d. of the matrix $C(A_1, B_1)$ is given by*

$$C(A_1, B_1) = [\,\Sigma\,|\,0\,]\, Q_1$$

for an orthogonal matrix Q_1.

PROOF:

$$C(A_1,B_1) = P^{-1}C(A,B)\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix} =$$

$$= P^{-1}X^t\,[\,\Sigma\,|\,0\,]\,Y\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix} =$$

$$= [\,\Sigma\,|\,0\,]\,Y\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}.$$

It suffices to consider $Q_1 = Y\mathrm{diag}\,(Q, \ldots, Q)$.

LEMMA 3. *For a given pair $(A,B) \in \mathcal{C}$ there exist an orthogonal matrix P, and an orthogonal matrix Q such that*

$$A_1 = P^{-1}AP = \begin{pmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{pmatrix}, \quad B_1 = P^{-1}BQ = \begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \end{pmatrix},$$

where $\overline{A}_1 \in M_r(\mathbf{R})$, $\overline{B}_1 \in M_{r\times m}(\mathbf{R})$, $1 \le r \le n-1$, with

$$\|\overline{A}_3\|_2 \le \|A_c\|_2 \frac{\sigma_{r+1}}{\sigma_r}, \quad \text{and} \quad \|\overline{B}_2\|_2 \le \sigma_{r+1}.$$

Here A_c denotes the companion matrix for A, that is to say,

$$A_c = \begin{pmatrix} 0 & \ldots & 0 & -\alpha_n \\ 1 & & 0 & -\alpha_{n-1} \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 1 & -\alpha_1 \end{pmatrix}$$

where α_i *are such that* $\det(tI-A)=t^n+\alpha_1 t^{n-1}\ldots+\alpha_n$.

PROOF: After Lemma 2, the matrix B_1 is given by

$$B_1=[\,\Sigma\,|\,0\,]\,Q_1\begin{pmatrix} I_m \\ 0 \end{pmatrix}.$$

Now partitioning the matrices

$$Q_1\begin{pmatrix} I_m \\ 0 \end{pmatrix}, \quad \text{and} \quad [\,\Sigma\,|\,0\,]$$

in the following manner

$$Q_1\begin{pmatrix} I_m \\ 0 \end{pmatrix}=\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

where $S_1\in M_{r\times m}(\mathbf{R})$, $S_2\in M_{(n-r)\times m}(\mathbf{R})$, $S_3\in M_{(m-1)n\times m}(\mathbf{R})$, and

$$[\,\Sigma\,|\,0\,]=\begin{pmatrix} \sigma_1 & & & & & & 0 & \ldots & 0 \\ & \ddots & & & & & & & \\ & & \sigma_r & & & & & & \\ & & & \sigma_{r+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \sigma_n & 0 & \ldots & 0 \end{pmatrix}=\begin{pmatrix} \Sigma_r & 0 & 0 \\ 0 & \Sigma_{\overline{r}} & 0 \end{pmatrix},$$

Then the matrix B_1 can be written as

$$B_1=\begin{pmatrix} \Sigma_r & 0 & 0 \\ 0 & \Sigma_{\overline{r}} & 0 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}=\begin{pmatrix} \Sigma_r S_1 \\ \Sigma_{\overline{r}} S_2 \end{pmatrix}=\begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \end{pmatrix},$$

with $\overline{B}_1\in M_{r\times m}$.

$$\|\overline{B}_2\|_2=\|\Sigma_{\overline{r}}S_2\|_2\leq\|\Sigma_{\overline{r}}\|_2\cdot\|S_2\|_2$$

Taking into account that $\Sigma_{\overline{r}}$ is a positive diagonal matrix, we have

$$\|\Sigma_{\overline{r}}\|_2=\sigma_{r+1}$$

and we have

$$\left\|\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}\right\|_2\leq\|Y\|_2\cdot\left\|\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}\right\|_2\cdot\left\|\begin{pmatrix} I_m \\ 0 \end{pmatrix}\right\|_2=1\cdot 1\cdot 1=1.$$

Then

$$\|S_2\|_2 \leq \left\| \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \right\|_2 \leq 1$$

and

$$\|\overline{B}_2\|_2 \leq \sigma_{r+1} \cdot 1 = \sigma_{r+1}.$$

To obtain a bound for $\|\overline{A}_3\|_2$ notice that if we consider

$$A_c \otimes I_m = \begin{pmatrix} 0 & \ldots & 0 & -\alpha_n I_m \\ I_m & & 0 & -\alpha_{n-1} I_m \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & I_m & -\alpha_1 I_m \end{pmatrix}$$

we have

$$\begin{aligned} (B \quad AB \quad \ldots \quad A^{n-1}B)(A_c \otimes I_m) &= (AB \quad \ldots \quad A^n B) \\ &= A(B \quad AB \quad \ldots \quad A^{n-1}B). \end{aligned}$$

The matrices A and A_1 have the same companion matrix, so

$$(B_1 \quad A_1B_1 \quad \ldots \quad A_1^{n-1}B_1)(A_c \otimes I_m) = A_1 (B_1 \quad A_1B_1 \quad \ldots \quad A_1^{n-1}B_1).$$

But by Lemma 2 we have that

$$[\Sigma \,|\, 0]\, Q_1(A_c \otimes I_m) = A_1\, [\Sigma \,|\, 0]\, Q_1$$

and

$$A_1 = [\Sigma \,|\, 0]\, Q_1(A_c \otimes I_m)([\Sigma \,|\, 0]\, Q_1)^+$$

where $+$ denotes the Moore-Penrose inverse. That is to say,

$$\begin{aligned} A_1 &= [\Sigma \,|\, 0]\, Q_1(A_c \otimes I_m)Q_1^t\, [\Sigma \,|\, 0]^+ = \\ &= \begin{bmatrix} \Sigma_r & & 0 \\ & \Sigma_{\overline{r}} & 0 \end{bmatrix} Q_1(A_c \otimes I_m)Q_1^t \begin{bmatrix} \Sigma_r^{-1} & \\ & \Sigma_{\overline{r}}^{-1} \\ 0 & 0 \end{bmatrix} = \\ &= \begin{pmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{pmatrix}. \end{aligned}$$

We denote the upper left $n \times n$ submatrix of $Q_1(A_c \otimes I_m)Q_1^t$ by Z and partition it in four blocks $\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$, with $Z_1 \in M_r(\mathbf{R})$.

Then

$$\begin{pmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{pmatrix} = \begin{pmatrix} \Sigma_r Z_1 \Sigma_r^{-1} & \Sigma_r Z_2 \Sigma_{\overline{r}}^{-1} \\ \Sigma_{\overline{r}} Z_3 \Sigma_r^{-1} & \Sigma_{\overline{r}} Z_4 \Sigma_{\overline{r}}^{-1} \end{pmatrix}.$$

Taking into account that

$$\|\Sigma_r^{-1}\|_2 = \sigma_r^{-1} \quad \text{and} \quad \|Q_1(A_c \otimes I_m)Q_1^t\|_2 = \|A_c \otimes I_m\|_2 = \|A_c\|_2,$$

we have

$$\|\overline{A}_3\|_2 = \|\Sigma_{\overline{r}} Z_3 \Sigma_r^{-1}\|_2 \leq \|\Sigma_{\overline{r}}\|_2 \cdot \|Z_3\|_2 \cdot \|\Sigma_r^{-1}\|_2 = \sigma_{r+1}\|Z_3\|_2 \sigma_r^{-1}.$$

But

$$\|Z_3\|_2 \leq \|A_c \otimes I_m\|_2 = \|A_c\|_2,$$

then

$$\|\overline{A}_3\|_2 \leq \|A_c\|_2 \frac{\sigma_{r+1}}{\sigma_r}.$$

THEOREM. *For a given pair $(A,B) \in \mathcal{C}$ we have*

$$\mu_{\mathbf{C}}(A,B) \leq \mu_{\mathbf{R}}(A,B) \leq \left(1 + \frac{\|A_c\|_2}{\sigma_r}\right)\sigma_{r+1}.$$

PROOF: We consider $(A_1 + \delta A_1, B_1 + \delta B_1)$ with

$$\delta A_1 = \begin{pmatrix} 0 & 0 \\ -\overline{A}_3 & 0 \end{pmatrix}, \qquad \delta B_1 = \begin{pmatrix} 0 \\ -\overline{B}_2 \end{pmatrix}$$

the pair $A_1 + \delta A_1, B_1 + \delta B_1$ is an uncontrollable pair of matrices for all $1 \leq r \leq n-1$, then

$$\|\delta A_1, \delta B_1\|_2 \geq \mu_{\mathbf{R}}(A_1, B_1) = \mu_{\mathbf{R}}(A,B).$$

Finally, in this case we have

$$\|\delta A_1, \delta B_1\|_2 \leq \|\delta A_1\|_2 + \|\delta B_1\|_2 = \|\overline{A}_3\|_2 + \|\overline{B}_2\|_2 \leq \|A_c\|_2 \frac{\sigma_{r+1}}{\sigma_r} + \sigma_{r+1} =$$
$$= \sigma_{r+1}\left(\frac{\|A_c\|_2}{\sigma_r} + 1\right).$$

Taking $r = 1, \ldots, r = n-1$, we obtain the following Corollary.

COROLLARY. *With the same notations as above we have*

$$\mu_{\mathbf{C}}(A,B) \leq \mu_{\mathbf{R}}(A,B) \leq \min_{1 \leq r \leq n-1} \left(\left(1 + \frac{\|A_c\|_2}{\sigma_1}\right)\sigma_2, \ldots, \left(1 + \frac{\|A_c\|_2}{\sigma_{n-1}}\right)\sigma_n\right).$$

EXAMPLE:

Let $(A, B) \in \mathcal{C}$ be the pair of matrices given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The controllability matrix is

$$C(A, B) = \begin{pmatrix} 0 & 3 & 15 \\ 0 & 0 & -3 \\ 1 & 4 & 16 \end{pmatrix}.$$

The singular values of $C(A, B)$ are

$$\sigma_1 = 22.68945837, \quad \sigma_2 = 1.018190280, \quad \sigma_3 = 0.3895735536.$$

The companion matrix of A is

$$A_c = \begin{pmatrix} 0 & 0 & 18 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}$$

and

$$\|A_c\|_2 = 21.93916272,$$

$$\sigma_2 \left(\frac{\|A_c\|_2}{\sigma_1} + 1 \right) = 2.002711015,$$

$$\sigma_3 \left(\frac{\|A_c\|_2}{\sigma_2} + 1 \right) = 8.783797845.$$

Then

$$\mu_{\mathbf{R}}(A, B) \leq 2.002711015.$$

If we consider $\delta A = 0$ and $\delta B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, the pair of matrices $(A + \delta A, B + \delta B)$ is obviously uncontrollable $(\operatorname{rank} C(A + \delta A, B + \delta B) = 0)$, and $\|\delta A, \delta B\| = 1 < 2.002711015$.

(II.2) μ-distance and tangent space

The s.v.d. characterization of the dimension of $\mathcal{O}(A, B)$ leads to the following Theorem.

THEOREM. *For a given controllable pair of matrices* (A,B) *with* $\dim \mathcal{O}(A,B)=a$ *a lower bound on the distance to the closest pair* $(A+\delta A, B+\delta B)$ *with* $\dim \mathcal{O}(A+\delta A, B+\delta B)=a-b$ *and* $b\geq 1$ *is given by*

$$\|(\delta A,\delta B)\|_F \geq \frac{1}{\sqrt{2n+m}}(\sum_{i=a-b+1}^{a}\sigma_i^2(\mathbf{T}))^{1/2} \geq \frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n+m}}.$$

PROOF: Let $(A+\delta A, B+\delta B)$ be a perturbed pair of matrices with

$$\dim T\mathcal{O}(A+\delta A, B+\delta B)=a-b.$$

Then

$$\operatorname{rank}(\mathbf{T}+\delta\mathbf{T})<a,$$

where

$$\delta\mathbf{T}=\begin{pmatrix}\delta A\otimes I_n - I_n\otimes \delta A^t & \delta B\otimes I_n & 0\\ -I_n\otimes\delta B^t & 0 & \delta B\otimes I_m\end{pmatrix},$$

$$\|(\delta A,\delta B)\|_F \leq \|\delta\mathbf{T}\|_F \leq \sqrt{2n+m}\|(\delta A,\delta B)\|_F.$$

The Eckart-Young and Mirsky Theorem for finding the closest matrix of a given rank (see [**6**]), gives that the size of the smallest perturbation in Frobenius norm that reduces the rank in $\mathbf{T}$ from a to $a-b$ with $b\geq 1$, is

$$\left(\sum_{i=a-b+1}^{a}\sigma_i^2(\mathbf{T})\right)^{1/2}.$$

Moreover if $\operatorname{rank}\mathbf{T}=a$, $\sigma_{a+1}(\mathbf{T})=\ldots=\sigma_{n^2+nm}(\mathbf{T})=0$,
Then,

$$\|(\delta A,\delta B)\|_F \geq \frac{1}{\sqrt{2n+m}}\left(\sum_{i=a-b+1}^{a}\sigma_i^2(\mathbf{T})\right)^{1/2} \geq \frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n+m}}.$$

As we say in (I.2), if $m=1$ any controllable pair of matrices is structurally stable. The we can deduce the following Corollary.

COROLLARY. *If* (A,B) *is a stable pair of matrices in* $M_n(\mathbf{C})\times M_{n\times 1}(\mathbf{C})$ *the distance to the closest uncontrollable pair* $(A+\delta A, B+\delta B)$ *is given by*

$$\|(\delta A,\delta B)\|_F \geq \frac{1}{\sqrt{2n+1}}(\sum_{i=n^2+n-b+1}^{n^2+n}\sigma_i^2(\mathbf{T}))^{1/2} \geq \frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n+1}}.$$

Example:

Let $(A,B)\in\mathcal{C}$ be the pair of matrices given by

$$A=\begin{pmatrix}0&8&0\\0&0&8\\0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0\\0\\8\end{pmatrix}.$$

The tangent space of $\mathcal{O}(A,B)$ is given by the matrix

$$\mathbf{T}=\begin{pmatrix}
0&0&0&8&0&0&0&0&0&0&0&0&0\\
-8&0&0&0&8&0&0&0&0&0&0&0&0\\
0&-8&0&0&0&8&0&0&0&0&0&0&0\\
0&0&0&0&0&0&8&0&0&0&0&0&0\\
0&0&0&-8&0&0&0&8&0&0&0&0&0\\
0&0&0&0&-8&0&0&0&8&0&0&0&0\\
0&0&0&0&0&0&0&0&0&8&0&0&0\\
0&0&0&0&0&0&-8&0&0&0&8&0&0\\
0&0&0&0&0&0&0&-8&0&0&0&8&0\\
0&0&-8&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-8&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-8&0&0&0&8
\end{pmatrix}$$

the smallest singular value of $\mathbf{T}$ is $\sigma_{\min}=3.560334942$.

Then
$$\frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n+m}}=1.345680121.$$

If we consider the pencil $\mathrm{A}+\lambda\mathrm{B}$ where

$$\mathrm{A}=\begin{pmatrix}A & B\end{pmatrix},\quad \text{and}\quad \mathrm{B}=\begin{pmatrix}I_3 & 0\end{pmatrix}$$

After [**4**], the tangent space of the orbit of this pencil is given by the matrix

$$T=\begin{pmatrix}A^t\otimes I_3 & -I_4\otimes \mathrm{A}\\ B^t\otimes I_3 & -I_4\otimes \mathrm{B}\end{pmatrix}$$

and
$$\frac{\sigma_{\min}(T)}{\sqrt{2n+m}}=0.1376763537.$$

Then, as we say in the Introduction we obtain a larger safety neigbourhood.

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